Mathematical Induction 1

1. Prove that $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$.

Let P(n) be the proposition : $2 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 2$

For P(1), L.H.S.$= 2 = 2^{1+1} - 2 =$R.H.S., $\therefore$P(1) is true.

Assume P(k) is true for some natural number k, that is,

$2 + 2^2 + 2^3 + \cdots + 2^k = 2^{k+1} - 2$.........(1)

For P(k + 1), $2 + 2^2 + 2^3 + \cdots + 2^k + 2^{k+1} = (2 + 2^2 + 2^3 + \cdots + 2^k) + 2^{k+1}$

$= (2^{k+1} - 2) + 2^{k+1}$, by (1)

$= 2 \times 2^{k+1} - 2$

$= 2^{k+2} - 2$

$\therefore$ P(k + 1) is true.

$\therefore$ By the Principle of Mathematical Induction, P(n) is true for all natural numbers, n.

2. Prove by Mathematical Induction: $\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$, for all integers $n \geq 2$.

Let P(n) be the proposition : $\sum_{i=1}^{n-1} i(i+1) = \frac{n(n-1)(n+1)}{3}$, for all integers $n \geq 2$.

For P(2), L.H.S.$= 1(1+1) = 2 = \frac{2\times(2-1)\times(1+1)}{3}$=R.H.S., $\therefore$P(2) is true.

Assume P(k) is true for some integerk,$k \geq 2$. that is,$\sum_{i=1}^{k-1} i(i+1) = \frac{k(k-1)(k+1)}{3}$......(1)

For P(k + 1), $\sum_{i=1}^{k} i(i+1) = \sum_{i=1}^{k-1} i(i+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1)$,by (1)

$= \frac{k(k+1)}{3}[(k-1) + 3] = \frac{k(k+1)}{3}(k+2)$

$= \frac{(k+1)[(k+1)-1][(k+1)+1]}{3}$

$\therefore$ P(k + 1) is true.

By the First Principle of Mathematical Induction, P(n) is true for integers $n \geq 2$.

3. Prove $1^3 + 3^3 + 5^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2 + 4n + 1)$ by mathematical induction.

Let $P(n)$: $1^3 + 3^3 + 5^3 + \cdots + (2n+1)^3 = (n+1)^2(2n^2 + 4n + 1)$

For $P(1)$, L.H.S. $= 1^3 + 3^3 = 28$, R.H.S. $= (1+1)^2[2(1^2) + 4(1) + 1] = 28$

$\therefore P(1)$ is true.

Assume $P(k)$ is true for some $k \in \boldsymbol{N}$, that is

$1^3 + 3^3 + 5^3 + \cdots + (2k+1)^3 = (k+1)^2(2k^2 + 4k + 1)$ ... (1)

For $P(k+1)$,

$1^3 + 3^3 + 5^3 + \cdots + (2k+1)^3 + (2k+3)^3$

$= (k+1)^2(2k^2 + 4k + 1) + (2k+3)^3 \ldots (2)$, by (1)

Now, writing $u = k + 1$, then

$(k+1)^2 = u^2$

$2k^2 + 4k + 1 = 2(u-1)^2 + 4(u-1) + 1 = 2u^2 - 1$

$(2k+3)^3 = [2(u-1)+3]^3 = (2u+1)^3 = 8u^3 + 12u^2 + 6u + 1$

By (2), $1^3 + 3^3 + 5^3 + \cdots + (2k+1)^3 + (2k+3)^3$

$= u^2(2u^2 - 1) + (8u^3 + 12u^2 + 6u + 1)$

$= (2u^4 - u^2) + (8u^3 + 12u^2 + 6u + 1)$

$= 2u^4 + 8u^3 + 11u^2 + 6u + 1$

$= (u+1)^2(2u^2 + 4u + 1)$ (use Factor Theorem)

$\therefore P(k+1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N}$.

4. Prove $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$by mathematical induction.

Let $P(n)$: $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$

For $P(1)$, L.H.S. $=1^2 = 1$, R.H.S. $=\frac{1}{3} \times 1 \times 1 \times 3 = 1$, $\therefore P(1)$ is true.

Assume $P(k)$ is true for some $k \in \boldsymbol{N}$, that is

$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1) \ldots (1)$

For $P(k+1)$,

$1^2 + 3^2 + 5^2 + \cdots + (2k-1)^2 + (2k+1)^2$

$= \frac{1}{3}k(2k-1)(2k+1) + (2k+1)^2$, by (1)

$= \frac{1}{3}(2k+1)[k(2k-1) + 3(2k+1)]$

$= \frac{1}{3}(2k+1)[2k^2 + 5k + 3]$

$= \frac{1}{3}(2k+1)(k+1)(2k+3)$

$= \frac{1}{3}\ (k+1)[2(k+1) - 1][2(k+1) + 1]$

$\therefore P(k+1)$ is true.

By the principle of mathematical induction, $P(n)$ is true for all $n \in \boldsymbol{N}$.

5. Prove $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$ by mathematical induction.

Let $P(n): 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

For $P(1)$, L.H.S. = $1^3 = 1$, R.H.S. = $\left[\frac{1\times 2}{2}\right]^2 = 1$

$\therefore P(1)$ is true.

Assume $P(k)$ is true for some $k \in \boldsymbol{N}$, that is

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \left[\frac{k(k+1)}{2}\right]^2 \quad \ldots (1)$$

For $P(k+1)$,

$$1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3$$

$$= \left[\frac{k(k+1)}{2}\right]^2 + (k+1)^3 \quad , \qquad \text{by } (1)$$

$$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right]$$

$$= \frac{(k+1)^2}{4}(k+2)^2$$

$$= \left[\frac{(k+1)[(k+1)+1]}{2}\right]^2$$

$\therefore P(k+1)$ is true.

By the Principle of mathematical induction, $P(n)$ is true $\forall n \in \boldsymbol{N}$.

Yue Kwok Choy

29/12/2020